Three is a crowd: A new cooperative solution concept for three-player games that takes parallel, bilateral negotiations to the center stage*

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PRELIMINARY

Abstract

We propose and analyze a new solution concept, the $R-$solution, for three-person, cooperative games. In the spirit of the Nash Bargaining Solution, the solution is founded on the analysis of the parallel, two-party negotiations that would be the alternative to the grand coalition. We compare the predictions of the $R-$solution with those of the Shapley value, and also show that the $R-$solution belongs to the Core whenever the latter is not empty. Finally, we discuss how the $R-$solution changes important conclusions of the Industrial Organization literature that deals with renegotiation in incomplete contracting frameworks.

KEYWORDS: cooperative games, simultaneous negotiations, $R-$Solution, incomplete contracts.

JEL classification numbers: C71, C78, L14.

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1 Introduction

Consider the following problem. A buyer, B, can trade with either of two possible sellers, S and E. These sellers may have different production costs, so that the gains from each possible, alternative trade are respectively 1 and $\alpha, \alpha \in [0,1]$. How would economic theorists model such a problem? How can we predict its outcome? There may be different answers to these questions, in particular with respect to the protocol used in negotiations or even with respect to the need of specifying one. However, it seems quite safe to say that most would predict trade between B and S if $\alpha < 1$ (efficiency) and also that E, the non-participating player, will appropriate zero surplus.

The example above describes what in essence is a problem of two interrelated, simultaneous, two-party negotiations. It also describes one of the simplest three-person (B, S, and E) cooperative games. We do not have a general model for the former, but we do have one for the latter. If we let v denote the characteristic function of the game, we could set $v(\{B,S,E\}) = v(\{B,S\}) = 1$, and $v(\{B,E\}) = \alpha$, where $v(Z)$ represents the value of coalition Z, and for all other coalitions (including one-player coalitions) $v(Z) = 0$. Thus, we may argue that instead of proposing *ad hoc*, non cooperative protocols to complete the description of the problem, or combining cooperative and non-cooperative elements to that effect, it would be more appropriate to use a solution concept from cooperative game theory. The most prominent candidate would probably be the Shapley value.

Unfortunately, the Shapley value predicts that E obtains a payoff equal to $\frac{\alpha}{6}$ (positive!).[1] Why is this counterintuitive outcome predicted? There are several ways to define (or characterize) the Shapley value, but perhaps the most popular is the "random order of arrival" story. Here, players arrive in a (uniformly distributed) random order. As players arrive, they form

[1] Generically, *probabilistic* values (see Weber 1988) would predict positive payoffs for E.

coalitions with the players already present after securing a payoff equal to the marginal contribution to that coalition. Then the Shapley value is equal to the expected vector of payoffs. In our example, there are six possible orders, and in one of them player E arrives second only to B. The marginal contribution of E to the coalition $\{B, E\}$ equals α, and that explains E's positive payoff. The Shapley value implicitly presumes that players B and E can credibly threat player S with trading among themselves leaving him out of the deal. Moreover, it also presumes that in that event E would be able to capture a positive share of α. Of course, these can be thought of as only threats that determine E's relative bargaining position, since eventually the efficient trade is predicted to take place. In other words, E's positive payoff may be thought of as a sort of bribe that allows B and S to implement the efficient trade without any interference from E.

In the problem we are considering, however, as well as in many other economic examples, we have no reason to assume that two-player coalitions are formed (and solidified) in any order, random or not, before the final trade is agreed upon. We would argue that it seems more natural to describe the problem for B, S, and E as one of simultaneous negotiations between B and S, on the one hand, and B and E, on the other. Even then, the negotiation between B and S might be affected by the alternative potential trade between B and E, and vice versa. However, *both* negotiations end *only* when one of the pairs reach an agreement and thus determine the final outcome. In this case, under what circumstances trade between B and E represents a meaningful alternative to the trade between B and S, and then can influence the terms of the latter? Also, if it does, under what circumstances E might be able to get a positive share of their surplus, α?

Consider the extreme case of $\alpha = 0$. This is equivalent to erase E out of the picture. Thus, we may feel comfortable using the Nash Bargaining Solution (NBS) applied to the negotiations between B and S and predicting (unless we have reasons to believe that players are heterogenous in their bargaining skills) that B and S will split the surplus equally. Consider now the case $\alpha \in \left(0, \frac{1}{2}\right)$. We still argue that the potential trade between B and E is not relevant. Indeed, suppose that B claims a payoff higher than $\frac{1}{2}$ in her negotiations with S since her fall-back option, a fraction of α, is higher than S's fall-back option, which is 0. In this case S can rightfully consider that such differences are not relevant since trade between B and E will not be reasonable to materialize. Indeed, in such trade B could at most obtain a payoff of $\alpha < \frac{1}{2}$. Thus, there is no reason to expect that B will ever refuse a payoff of $\frac{1}{2}$ due to a more attractive deal with E. In other words, if $\alpha < \frac{1}{2}$, we should predict that the efficient trade will take place and that players B, S and E will get $\frac{1}{2}, \frac{1}{2}$ and 0, respectively.

Suppose now that $\alpha \in \left[\frac{1}{2}, 1\right]$. In this case the situation is quite different. An agreement of B and S to split the surplus equally will leave B with feasible, alternative trades at mutual advantage with E. Thus, potential deals between B and E are now relevant even if they do not actually occur (with positive probability). So, even if it is only reasonable to predict that the efficient trade must take place, how will B and S split the surplus? Could we predict that B gets a payoff strictly higher than α? If so, we would be back in a position analogous to the split $50 - 50$ when $\alpha < \frac{1}{2}$. Indeed, if this is the predicted deal, then any mutually advantageous deal between B and E would leave B worse off than the predicted outcome in the negotiations between B and S, and therefore negotiations between B and E would again be irrelevant. But if this is the case, then there is no reason for B to expect more than 50% of the surplus. Thus, predicting any split where B appropriates more than α will not be consistent. On the other hand, suppose that we predict that player B gets a payoff strictly lower than α in her negotiations with S. If that was so case, there would be mutually advantageous negotiations between her and E, and we should then instead predict that some of these deals will be stricken instead, a deal that again could be improved upon by B and S, etc. Thus, the only reasonable prediction is that the efficient trade takes place and players B, S and E obtain $\alpha, 1-\alpha$, and 0, respectively. Thus, when $\alpha \in \left[\frac{1}{2}, 1\right]$ the alternative trade between B and E is relevant to determine the outcome of the negotiations, and nevertheless player E does not obtain any share of the surplus. Or, put in other words, the *non-participating* player obtains nothing and the participating, non *indispensable* player obtains a payoff equivalent to his *competitive advantage*, $1-\alpha$. In the limit as α approaches one (the *gloves game*), and both non indispensable players become perfect substitutes then the indispensable player is able to appropriate the entire surplus (in analogy with Bertrand competition).

In this paper we develop a solution concept, the $R-$solution, for three-player cooperative games that in particular is consistent with these predictions for simultaneous, alternative two-player negotiations. The concept is in the spirit of the NBS and treats the two possible trades symmetrically. From this point of view, the $R-$solution sets "disagreement points" in the two-player negotiations endogenously. Moreover, unlike the Shapley value, it has the property that ex-post, that is, after the deal is stricken, there are no deviations by two of the parties that are profitable for the deviating parties without the concourse of the other player. In this sense, the solution we have outlined above does not need to preclude any coalition from exploring opportunities before a final deal is stricken. In other words, when parties divide the surplus they need not envision any coalition to form and stick together before this deal is reached. This seems an appropriate property of a solution when negotiations between all three parties are simultaneous and only one trade is feasible.

Although the motivation for our concept comes from alternative trades between one buyer and two sellers (or two buyers and one seller), many economic issues and models that share its key elements (see Section 4) require that we go beyond this simple problem and define a full fledged solution concept for general, three-player games. In order to do so, we will need generalizing the problem in two ways. First, we should allow the coalition of players S and E to generate positive surplus, $v(B,E) > 0$: player B is not indispensable anymore, but her contribution is necessary to achieve the efficient outcome. In fact, once we do that we remove the asymmetry in the roles of our buyer and sellers. Second, we should allow the grand coalition to generate net positive value: $v(B,S,E) > v(B,S)$.

Thus, let us assume that $v(\{B,S,E\}) = v(\{B,S\}) = 1$, $v(\{B,E\}) = \alpha$, and $v(B,E) = \beta$ (and still assume that the value of each individual player is 0). Without loss of generality assume that $1 \geq \alpha \geq \beta \geq 0$. First, note that the discussion before for the case $\alpha < \frac{1}{2}$ still indicates that if ($\beta \leq$) $\alpha < \frac{1}{2}$ then alternative trades for both sides, B and S, are irrelevant and we should still predict that B and S will trade and split the surplus equally. Second, if $\alpha \geq \frac{1}{2}$ and $\beta < 1 - \alpha$ then again the outcome will be the one obtained for $\beta = 0$: the alternative trade for B now matters but not the alternative trade for S.

The situation changes when $\beta + \alpha \geq 1$. In this case, all three bilateral negotiations are relevant, and all three trades could conceivably take place. There is no deal between two-parties that is such that none of them can explore more attractive, mutually advantageous deals with the party left out. Thus, all three trades may be expected with positive probability. But for that to be a consistent prediction, it must be the case that no party prefers one of her possible trades to the other. Indeed, consider any trade in particular. This trade necessarily leaves one party outside, and therefore eager to see either of the alternative trades materialize. Thus, if one of the parties to the trade considered does prefer dealing with the excluded party to taking part in the trade then we should never expect the trade to take place. Therefore, if we denote by u_i player i's predicted payoff, $i = B, S, E$, if an agreement is reached with either of i's potential partners, then it must be the case that:

$$\begin{aligned} u_B + u_S &= 1 \\ u_B + u_E &= \alpha \\ u_S + u_E &= \beta \end{aligned}$$

This system of equations determines all payoffs in "bilateral trade". As we have mentioned, all these trades are conceivable now. We need to specify the probability that each of them would materialize. It tuns out that, for each vector of parameter values (α, β), there is a unique probability distribution such that (u_i, u_j) is in fact the NBS in the bilateral negotiation between i and j, assuming that their disagreement points are the expected (according to that distribution) payoff in their respective alternative trades. That is, there is only one probability distribution compatible with Nash behavior and consistent with players' indifference between their alternative trade opportunities.

For the first time, it is now important to consider what the coalition of three players can achieve. Indeed, when $\beta + \alpha \geq 1$ there is room to form a genuine grand coalition. Since, as we have mentioned, all three trades can conceivably materialize if parties insist on decentralized negotiations, two-party negotiations by themselves do not guarantee that the outcome will be efficient, in the sense that the payoff of the grand coalition, 1, is achieved. Thus, since there are gains from conducting three-party negotiations, we should expect these to take place. Moreover, having analyzed the two-party negotiations and then possessing a probabilistic assessment of which one of them would materialize if the three-party negotiation were to fail, we can easily compute the "disagreement point" for such negotiation. Thus, once again in the spirit of the NBS, we presume that an agreement between the three players will eventually take place, it will include the efficient trade, and will dictate a division an equal, three-way split of the surplus net of the disagreement payoffs computed as the expected payoffs in two-party negotiations.

As mentioned above, a standard interpretation of the Shapley value justifies the positive payoff of the non-participating player, E, on the basis of his contribution to facilitate the efficient trade. We consider this a very useful insight. In this respect, our contribution is to clarify the set of circumstances under which the non-participating player does play an effective role, and therefore needs to be taken on board to facilitate the efficient trade.

Finally, when $v(B, S, E) > v(B, S)$ forming the grand coalition adds additional surplus $v(B, S, E) - v(B, S)$. Our solution concept extends without new complications to that situation by predicting that this additional surplus is equally split like the surplus of the three-party negotiation that we have just commented.

Summarizing, in the next section we will propose a new solution concept for three-player games in which the payoffs are the NBS of the three-player negotiation with disagreement points equal to the expected utility of the outcome of the simultaneous two-player negotiations; which, in turn, consists of a vector of predicted payoffs for each negotiation plus a probability distribution over these alternative trades. In each two-player negotiation the payoffs are the NBS with disagreement points equal to the expected payoff that each player obtains in their alternative trade.

2 The $R-$solution of a three-person game

Let $N = \{1, 2, 3\}$ be the set of players, and let 2^N represent the set of subsets of N. An element $Z \in 2^N$ represents a coalition. A game in characteristic form is the pair (N, v), where $v : 2^N \rightarrow R$ satisfies $v(\varnothing) = 0$. We assume v to be *monotone*.

Assumption 1 (monotonicity): If $Z \subset Z'$, then $v(Z) \leq v(Z')$.

To save some space, we will use an abbreviated notation for the v function. Thus, we will let $v_{ij} = v(\{i, j\})$, $v_i = v(\{i\})$ and $V = v(\{1, 2, 3\})$. Also, without loss of generality, we will assume that $v_{12} - v_1 - v_2 \geq v_{ij} - v_i - v_j$ for all $j \neq i$. That is, the coalition $\{1, 2\}$ is the (weakly) most "efficient" among the two-player coalitions. Likewise, we will assume that $v_{13} - v_1 - v_3 \geq v_{23} - v_2 - v_3$. Thus, the coalition $\{1, 3\}$ is the second most "efficient" coalition. Also, without loss of generality we will normalize $v_i = 0$ for all $i = 1, 2, 3$. Thus, all that we will obtain below relative to payoffs should be interpreted as payoffs in excess of these one-player coalitions' payoffs. Apart from this caveat, the normalization is without loss of generality. Also, every time we write "for all i, j" or "for all i, j, k" we mean for all $i, j = 1, 2, 3, i \neq j$, and for all $i, j, k = 1, 2, 3$, $i \neq j \neq k, i \neq k$, respectively. That is, different sub/superindeces in the same expression will always denote different players.

The heart of our solution concept is a prediction of the outcomes of the three possible bilateral negotiations, including which of these negotiations would succeed (with what probability), should three-player negotiations failed. As we discussed in the previous section, in many economic applications this is in fact all that will be needed to predict the outcome of the game. Thus, we begin by formally propose a solution concept for these simultaneous, two-party negotiations.

A solution for bilateral negotiations includes the fall-back option and the predicted payoff for each player i in each bilateral negotiation ij, which are denote by t_i^{ij} and u_i^{ij} respectively; and a probability distribution over these three negotiations, with p_{ij} denoting the probability that players i and j strike a deal. Given these fall-back options, t_i^{ij}, and in the spirit of the NBS, players i and j share any surplus equally, provided this surplus is positive ($v_{ij} \geq t_i^{ij} + t_j^{ij}$). That is, $u_i^{ij} = \frac{1}{2}\left(v_{ij} + t_i^{ij} - t_j^{ij}\right)$. However, if their fall-back options sum up to an amount in excess of the value of the coalition, $v_{ij} < t_i^{ij} + t_j^{ij}$, then players will not be willing to strike any deal (at least one of them prefers to have the chance to reach an agreement with its alternative partner). Therefore, the payoffs from this negotiation are zero (i.e., their individual payoffs).

In turn, the fall-back options are computed according to the payoffs predicted in, and the probability distribution over, alternative two-party negotiations. In particular, if the negotiation between i and j flounders, and players contemplate their options in the large picture of all two-player negotiations, and therefore what they expect to get as a default for this negotiation, t_i^{ij}, they can see that, (i) with probability p_{ij} what they face is precisely this default, t_i^{ij}, (ii) with probability p_{ik} coalition (i, k) will strike a deal, and player i's payoff is u_i^{ij}, and (iii) with probability p_{jk} it will be coalition (j, k) who will strike a deal, and hence i's payoff is zero. Thus, $t_i^{ij} = p_{ij} t_i^{ij} + p_{ik} u_i^{ik}$. If $p_{ij} < 1$ we can rewrite this expression as follows:

$$t_i^{ij} = \frac{p_{ik}}{1 - p_{ij}} u_i^{ik}$$

Thus, player $i's$ fall-back option in its negotiation with j is the expected payoff in its alternative negotiation, where this payoff is weighted by the "conditional" probability of being able to reach an agreement with player k, given that its negotiation with j has come to a halt.

So far, we have described how payoffs are determined for a given probability distribution. The description of our solution concept is completed by imposing one condition on how payoffs affect probabilities. In particular, we require that if $p_{ij} > 0$ then $u_i^{ij} \geq u_i^{ik}$ and $u_j^{ij} \geq u_j^{jk}$. That is, an agreement between players i and j is reached with positive probability only if both players weakly prefer such agreement over their alternatives.

Thus, the outcome of simultaneous, bilateral negotiations builds on the NBS for each negotiation with endogenous fall-back options. The solution sketched above would not be well (uniquely) defined if probability distributions were degenerate (in the above expression, if $p_{ij} = 1$). In order to overcome this difficulty and the multiplicity this would allow, we will define our solution concept as the limit of a sequence of some "restricted" outcomes. More specifically, we begin by defining a restricted solution concept.

Definition 1 *For $\epsilon > 0$, an $\epsilon - R-$solution to simultaneous, bilateral negotiations, $\epsilon - R$ SSBN for short, for the three-player game (N, v) is a triple $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}_{i,j=1,2,3}$ that satisfies:*

1)

$$u_i^{ij}(\epsilon) = \begin{cases} \frac{1}{2}\left(v_{ij} + t_i^{ij}(\epsilon) - t_j^{ij}(\epsilon)\right) & \text{if } v_{ij} \geq t_i^{ij}(\epsilon) + t_j^{ij}(\epsilon) \\ 0 & \text{otherwise;} \end{cases}$$

2) $t_i^{ij}(\epsilon) = p_{ij}(\epsilon)\, t_i^{ij}(\epsilon) + p_{ik}(\epsilon)\, u_i^{ik}(\epsilon)$, for all i, j, k;

Definition 2 *3) $p_{12}(\epsilon) + p_{13}(\epsilon) + p_{23}(\epsilon) = 1$; $p_{ij}(\epsilon) \leq 1 - \epsilon$ for all i, j; and for all i, j, k, $p_{ij}(\epsilon) > \epsilon$ only if $u_i^{ij}(\epsilon) \geq u_i^{ik}(\epsilon)$ and $u_j^{ij}(\epsilon) \geq u_j^{jk}(\epsilon)$.*

We now study the existence and limiting properties of this $\epsilon - R$ solution concept for simultaneous, bilateral negotiations.

Proposition 1 *For ϵ small, an ϵ-R SSBN, $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}$, exists for the game (N, v). Moreover, $\lim_{\epsilon\to 0}\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}$ exists (and then is unique). Also,*

1) if $v_{12} \geq v_{13} + v_{23}$ and $v_{13} \leq \frac{1}{2}v_{12}$, then $\lim_{\epsilon\to 0} p_{12}(\epsilon) = 1$, and $\lim_{\epsilon\to 0} u_1^{12}(\epsilon) = \lim_{\epsilon\to 0} u_2^{12}(\epsilon) = \frac{1}{2}v_{12}$;

2) if $v_{12} \geq v_{13} + v_{23}$ and $v_{13} \geq \frac{1}{2}v_{12}$, then $\lim_{\epsilon\to 0} u_1^{12}(\epsilon) = v_{13}$ whereas $\lim_{\epsilon\to 0} u_2^{12}(\epsilon) = v_{12} - v_{13}$, and if $v_{13} < v_{12}$ then $\lim_{\epsilon\to 0} p_{12}(\epsilon) = 1$; and

3) if $v_{12} \leq v_{13} + v_{23}$ then $\lim_{\epsilon\to 0} u_i^{ij}(\epsilon) = \lim_{\epsilon\to 0} u_i^{ik}(\epsilon) \equiv u_i = \frac{v_{ij}+v_{ik}-v_{jk}}{2}$, for all i, j, k, and $\lim_{\epsilon\to 0} p_{ij}(\epsilon) \equiv p_{ij} = \frac{u_i u_j}{u_1u_2+u_1u_3+u_2u_3}$.

Proof. See Appendix. ■

We now have the instrument needed to predict the outcome of two-player, simultaneous negotiations, and therefore the fall-back options in the three-player negotiation. Thus, we are ready to define our solution concept for the game (N, v), the $R-$solution.

Definition 3 *An $R-$solution for the three-player game in characteristic form (N, v) is a triple (U_1, U_2, U_3) that satisfies:*

a) $U_i = \frac{1}{3}(V + 2T_i - T_j - T_k)$ for all i, j, k, where

b) $T_i = p_{ij}u_i^{ij} + p_{ik}u_i^{ik}$, and

c) $p_{ij} = \lim_{\epsilon\to 0} p_{ij}(\epsilon)$, and $u_i^{ij} = \lim_{\epsilon\to 0} u_i^{ij}(\epsilon)$, where for each ϵ, the triple $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}_{i,j=1,2,3}$ is a $\epsilon - R$ SSBN for the game (N, v).

The grand coalition shares the surplus $V - T_1 - T_2 - T_3$ according to the NBS. Player $i's$ fall-back option, T_i, is her expected payoff in the simultaneous, bilateral negotiations. More specifically, $T_i = p_{ij}u_i^{ij} + p_{ik}u_i^{ik}$, where both probabilities and payoffs in each bilateral negotiation are the limit of the ϵ-R solution, as ϵ goes to 0.

Characterizing this solution, in particular its existence and uniqueness, requires characterizing $\lim_{\epsilon\to 0}\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}_{i,j=1,2,3}$. This is done in Proposition 1. Other than that, the $R-$solution is a three-player NBS. Since T_i, $i = 1, 2, 3$ exists and is unique, the proof of the following theorem is straightforward.

Theorem 1 *The $R-$solution exists and is unique.*

The computation of the $R-$solution is in fact extremely simple. What we offer below can be considered a user manual, and to that effect we take into account the value of one-player coalitions that so far we have normalized to 0. It is convenient to split the parameter space in three different regions (See Figure 1). Let Δ_{ij} be the net surplus that can be created by coalition (i, j), i. e. $\Delta_{ij} = v_{ij} - v_i - v_j$. In the previous section, we set these surpluses to 1, α and β, for $ij = 12$, 13, and 23 respectively. Recall that we are assuming, without loss of generality, that $\Delta_{12} \geq \Delta_{13} \geq \Delta_{23} \geq 0$. Define Region 1 as the area satisfying $\Delta_{13} \leq \frac{1}{2}\Delta_{12}$, Region 2 as the area satisfying $\Delta_{13} \geq \frac{1}{2}\Delta_{12}$ and $\Delta_{23} \leq \Delta_{12} - \Delta_{13}$, and Region 3 as the area satisfying $\Delta_{13} \geq \frac{1}{2}\Delta_{12}$ and $\Delta_{23} \geq \Delta_{12} - \Delta_{13}$. The following Table 1 contains the expression for the $R-$solution in each of these regions.

Table 1: The $R-$solution

	Region 1	Region 2	Region 3
U_1	$\frac{2V+v_{12}+3v_1-3v_2-2v_3}{6}$	$v_{13} - \frac{4v_3}{3} + \frac{V-v_{12}}{3}$	$\frac{V+v_{12}+v_{13}-2v_{23}}{3}$
U_2	$\frac{2V+v_{12}-3v_1+3v_2-2v_3}{6}$	$\frac{2(v_{12}-v_{13})}{3} + \frac{2}{3}v_3 + \frac{V-v_{13}}{3}$	$\frac{V+v_{12}+v_{23}-2v_{13}}{3}$
U_3	$v_3 + \frac{V-v_{12}-v_3}{3}$	$\frac{2}{3}v_3 + \frac{V-v_{12}}{3}$	$\frac{V+v_{13}+v_{23}-2v_{12}}{3}$

Computing these values for the first two regions is straightforward. Region 3 is a little more involved, since the expressions for T_i are also more involved. However, there is an interesting property that $R-$solution satisfies that will simplify these computations: for each game (N, v) there exists a number Ψ such that

$$p_{ij}\left(u_k - v_k\right) = \Psi \text{ for all } i, j, k. \tag{1}$$

The surplus that player k obtains if one of the coalitions in which he participates is called to reach an agreement is $u_k - v_k$. Also, p_{ij} is the probability that player k does not get $u_k - v_k$. Therefore, condition (1) indicates that the "loss" experienced by player i with respect to the benchmark where he is able to secure u_i with probability one, is the same for all $i = 1, 2, 3$. This property will drastically simplify the computation of final payoffs. More specifically, player $i's$ expected payoff in the solution of the bilateral negotiations is:

$$T_i = \left(p_{ij} + p_{ik}\right) u_i + \left(1 - p_{ij} + p_{ik}\right) v_i = v_i + \left(p_{ij} + p_{ik}\right)\left(u_i - v_i\right).$$

We can further rewrite this expression using condition (1):

$$T_i = v_i + (u_i - v_i) - \Psi = u_i - \Psi.$$

As a result the $R-$solution for player i is given by:

$$U_i = \frac{1}{3}(V + 2T_i - T_j - T_k) = \frac{1}{3}\left(V + 2u_i - u_j - u_k\right),$$

and making further use of Proposition 1 we obtain the final expression:

$$U_i = \frac{1}{3}\left(V + v_{ij} + v_{ik} - 2v_{jk}\right)$$

3 Discussion

In this section we discuss the properties of the $R-$solution and how it compares to other solution concepts like the Shapley value and the Core.

3.1 Payoff differentials between the Shapley value and the $R-$solution

The $R-$solution only coincides with the Shapley value at two points of the parameter space: $v_{13} = v_{23} = 0$ and $v_{13} = v_{23} = v_{12}$.[2] We first investigate how the $R-$solution treat different players with respect to the Shapley value. This comparison is straightforward and is summarized in the following proposition (See Figure 2).

Proposition 2 *With respect to the Shapley value, according to the $R-$solution:*

(i) Player 3's payoff is lower for all parameter values

(ii) Player 2's payoff is lower if and only if $v_{13} \geq \frac{3}{4}v_{12} - \frac{1}{4}v_{23}$ in Region 2, and $v_{13} \geq \frac{1}{2}\left(v_{12} + v_{23}\right)$ in Region 3.

(iii) Player 1's payoff is lower if and only if $v_{13} \geq 2v_{23}$ in Region 1, and $v_{13} \leq \frac{3}{5}v_{12} - \frac{2}{5}v_{23}$ in Region 2.

The intuition is somewhat transparent along the vertical axis $(v_{23} = 0)$. Suppose v_{13} is relatively low. Then, according to the $R-$solution the alternative trade between players 1 and 3 is irrelevant and, unlike the Shapley value, any increase in v_{13} is not reflected in higher payoffs for players 1 and 3 and a lower payoff for player 2. As a result player 2 is better off and players 1 and 3 worse off. Suppose now that v_{13} is relatively high. Player

[2] In the first point ($v_{13} = v_{23} = 0$) the $R-$solution coincides with the NBS of the game for players 1 and 2. In this sense, both the Shapley value and the $R-$solution are generalizations of the NBS to the case of three players.

3 is a close, although inferior, substitute of player 2. This does not allow player 3 to secure a positive payoff, but the competition effect of its presence drastically erodes player 2's payoff. Since the Shapley value treats players in a more "egalitarian" manner, in this region of the parameter space players 2 and 3 are worse off. Finally, for intermediate values of v_{13}, coalition $(1,3)$ becomes relevant, which benefits player 1, but player 2 still maintains a significant competitive advantage. As a result only player 3 is worse off.

If $v_{23} > 0$ then the balance between different effects is more complex. Even in Region 3, where the $R-$solution grants Player 3 a positive payoff, this is smaller than the one granted by the Shapley value. If v_{13} is sufficiently high player 2 is worse off unless v_{23} is sufficiently close to v_{13}. That is, whenever players 2 and 3 are close substitutes from the point of view of player 1, this tends to hurt player 2 unless their joint value, v_{23}, is sufficiently strong.

3.2 Which Shapley axiom is violated by the $R-$solution?

As is well known (see for instance Winter, 2002), the Shapley value is the only value that satisfies the axioms of efficiency, symmetry, dummy player, and additivity. That means that the $R-$solution should violate at least one of these axioms. The $R-$solution satisfies efficiency, that is, for any game $U_1 + U_2 + U_3 = V$. It also satisfies symmetry. That is, if U is the $R-$solution of (N, v) and U' is the$R-$solution of (N, v') where $v'(Z) = v(Z')$ and $Z' = \{i \in N \,|\mu(i) \in Z\}$, for some bijection $\mu : N \to N$ then $U_i = U'_{\mu(i)}$ for all $i \in N$. In other words, the name of the player has no effect on its value. Also, the $R-$solution satisfies the dummy axiom. In other words, if $v(S \cup i) - v(S) = 0$ for every $S \subset N$, then $U_i = 0$. Thus, the $R-$solution must violate the additivity axiom. That is, if (N, v) and (N, v') are two games with solutions U and U' respectively, and we consider the game (N, v'') where $v''(Z) = v(Z) + v'(Z)$ for all $Z \subset N$, it may be that its $R-$solution

U'' does not satisfy $U_i'' = U_i + U_i'$.

We will argue that for the class of problems that we are envisioning this is a strength of the concept rather that a weakness. Consider once again our initial example with one buyer, B, and two potential sellers, S and E. Suppose that there are two goods and the buyer demands one unit of each. In the production of the first, S has a cost advantage, so that $v(B,S) = 1$ and $v(B,E) = \alpha \in \left(\frac{1}{2}, 1\right)$, whereas in the production of the second it is E who has the cost advantage, so that $v'(B,E) = 1$ and $v'(B,S) = \alpha$. According to the R-solution, E obtains 0 in the first game and $1-\alpha$, in the second. The game $v'' = v + v'$ then satisfies that $v''(B,E) = v''(B,S) = 1+\alpha$, and $v(B,S,E) = 2$. Imposing additiviy means that player E, for instance, should still fetch $1-\alpha$ in game v''. In fact, in v'' the $R-$solution grants him one third of that amount. That is, additivity implies that the negotiations over the two goods are conducted independently, while the $R-$solution implicitly presumes that both negotiations are tied, which is very reasonable in this abstract setting. In particular, the fact that S may also supply the good for which he has a competitive disadvantage is a handicap. One may argue that this makes no sense, since E can always "destroy" his ability to supply that good. But consider such possibility, that is the game $\widetilde{v} = v''$ except that $\widetilde{v}(B,S) = 1$. This is a game in our Region 2, with E as player 3. If three-party negotiations fail, as we argued in the introduction, there seems to be no reason to expect that player 3 gets any payoff in this region. Thus, the most he can expect is in fact a third of $V - \widetilde{v}(B,E)$, that is, a third of $1-\alpha$. This is in fact what he gets in the $R-$solution.

3.3 The Core and the $R-$solution

It is well known that the Shapley value is not necessarily in the Core even when the Core is not empty. This is another difference between the

$R-$solution and the Shapley value or any probabilistic value.[3]

Proposition 3 *If the Core of the game* (N, v) *is not empty, then the* $R-$*solution* U *is in the Core.*

Proof. Since the $R-$solution is efficient, all we have to show is that either $U_i + U_j \geq v_{ij}$ for all i, j or the Core is empty. When $v_{12} \geq v_{13} + v_{23}$ this is satisfied trivially. Indeed, in that case $U_3 = 0$, $U_i \geq v_{i3}$, for $i = 1, 2$, and $U_1 + U_2 \geq v_{12}$. When $v_{12} < v_{13} + v_{23}$ the Core may be empty. Indeed, this is the case if, for instance, $V = v_{12}$. We first derive the condition for the Core not to be empty in this case. An element of the Core is a positive vector (x_1, x_2, x_3) such that: (i) $x_1 + x_2 + x_3 = V$ and (ii) $x_i + x_j \geq v_{ij}$ for all i, j. Adding up these last three conditions, we obtain $x_1 + x_2 + x_3 \geq \frac{v_{12}+v_{13}+v_{23}}{2}$, which combined with condition (i) gives:

$$V \geq \frac{v_{12} + v_{13} + v_{23}}{2}. \tag{2}$$

Thus, assume that condition (2) is satisfied and also that $v_{12} < v_{13} + v_{23}$. Then, it is immediate to check that $U_i + U_j \geq v_{ij}$ for all i, j. ■

4 Applications

In this section we illustrate the use of our solution concept by discussing three important examples in the literature where ex-post bargaining among three agents play a central role. We will discuss how modelling the outcome of these negotiations as the $R-$ solution of the game that agents play affect the conlusions of those papers. We will do that by introducing a common framework as follows. There are three goods, one consumption good and two inputs, and three players, B, S, and E. Thus, we recover the names of

[3] As shown by Weber (1988), a probabilistic value is efficient only if it is a *random-order value*, and in our superadditive setting efficiency is a condition for an allocation to be in the Core. The set of all random-order values contains the Core, but no single one is "always" contained in the Core even if we restrict attention to three player games.

players used in Section 1, but keep the notation introduced in Section 2 with only substituting B, S, and E for 1, 2, and 3. Player B is a consumer who is able to obtain a potential utility w from one unit of the consumption good. Player S can produce one unit of an input at a cost c_s. He also makes an investment decision, $x \in [0, 1]$, that costs $\Psi(x)$, which is a twice differentiable function with $\Psi(0) = \Psi'(0) = 0$, $\Psi'(x) > 0, \Psi''(x) > 0$, $\lim_{x \to 1} \Psi(x) = \infty$. Higher investment may either enhance player B's utility, w, or may reduce his own production costs, c_s. Player E can also produce one unit of the second input at cost c_e. Player E does not make any explicit investment decision but he is associated with a random variable $y \in [0, 1]$, which is distributed according to the cumulative function $H(y)$. The transformation of inputs into the consumption good may require the use of an asset.

There are three periods, $t = 0, 1, 2$. In period 0 agents sign contracts that in particular may assign property rights over the asset. In period 1 player S chooses x and simultaneously the realization of y becomes known. Hence, player S is uncertain about y when he chooses x. Both x and y are observable but not verifiable. Actual trade takes place in period 2, after investment has been chosen and uncertainty resolved. In most cases, there is room to (re)negotiate production decisions and how the gains from trade are shared. The common theme in the papers discussed below is how (incomplete) contracts signed in period 0 affect investment decisions in period 1, given that agents anticipate renegotiation in period 2. Thus, in each case we will be interested in analyzing the payoffs of S as a function of his investment x under different contracts, and compare his best decision to the efficient one. Each contract (or the lack of it) plus the realization of the uncertain y and the decision x determine a three-player cooperative game (negotiation) in period 2.

4.1 Allocation of property rights (Hart and Moore, 1990)

Hart and Moore (1990), HM, study how the allocation of property rights over assets affect the ex-post relative bargaining position of different players, which in turn determine ex-ante incentives to make asset-specific investments. In their introductory example, players S, E and B are called the chef, the skipper and the tycoon, respectively, and the asset is a yacht. If we let w be given by:

$$w(x, y) = x + y,$$

and $c_s(x) = c_e(y) = 0$, we have a version of their example[4]. In other words, the chef and the skipper provide their services on the yacht, which are enjoyed by the tycoon. The first best level of effort, x^*, is implicitly given by:

$$\Psi'(x^*) = 1.$$

Contracts in period 0 are incomplete, and in fact the only aspect that can be contracted is the right to use (or exclude from the use of) the asset.

One of the insights of this paper is that, even though player B does not take any efficiency-related decision, it may be optimal to allocate all residual rights to this player. The reason is that player B is indispensable to the asset (without her participation the yacht is useless). The authors arrive to this conclusion assuming that payoffs in period 2 are determined according to the Shapley value. Each potential assignment of property rights originates a different game in period 2. In all of them, the grand coalition has a value

$$V = x + y. \tag{3}$$

Also, in all of them any coalition that does not include B has a value of 0. (B is indispensable.)

[4] Hart and Moore assume that the investment x can take only two values, 0 or 100. Dealing with continuous variables is more illustrative but basically equivalent.

(i) Suppose, first, that player S (the chef) owns the yacht. This means that no coalition that excludes S generate any value. Thus,

$$v_{SB} = x, \tag{4}$$

and appart from the grand coalition, all other coalitions have value 0.

(ii) Suppose now that player E (the skipper) owns the asset. In this case the only coalition with a positive value other than the grand coalition is

$$v_{EB} = y. \tag{5}$$

(iii) Finally, suppose that player B (the tycoon) owns the asset. Now, there are two two-player coalitions with a positive value, which are given by equations (4) and (5)

In case (i), S's marginal contribution to the grand coalition (with weight $\frac{1}{3}$) is $x + y$, and his contribution to the coalition with player B (with weight $\frac{1}{6}$) is x. Thus, his Shapley value is

$$U_s = \frac{x}{2} + \frac{y}{3}.$$

Therefore, in period 1 player S chooses a level of effort $\overline{x}$, which is given by:

$$\Psi'(\overline{x}) = \frac{1}{2}.$$

Clearly, there is underinvestment, since $\overline{x} < x^*$.T here is an ex-post holdup problem, which is typical in this incomplete contract settings. Once investment is sunk, the player that paid for the investment is forced to share its returns with other players.

In case (iii), player S contributes x to the grand coalition, and also (with weight $\frac{1}{6}$) x to the coalition with player B. Thus,

$$U_s = \frac{x}{2}.$$

Therefore, player S is worse off than in the case he owns the asset, but investment incentives are identical in both cases. In particular, if player B

owns the asset, player S chooses in period 1 a level of investment equal to $\overline{x}$.

Finally, in case (ii), player S's contribution to the grand coalition (with weight $\frac{1}{3}$) is x, and thin his Shapley value is:

$$U_s = \frac{x}{3}.$$

As a result, the underinvestment problem is exacerbated, since in period 1 player S chooses $\underline{x} < \overline{x}$, which is given by $\Psi'(\underline{x}) = \frac{1}{3}$. From an efficiency point of view, in this example either player S or B should own the asset, but not player E. The asset should be in the hands of either the agent that exerts an effort or the agent which is indispensable for the asset.

Conclusion 1 *Under the Shapley value it is optimal that player B (the indispensable player) owns the asset.*

This celebrated result, however, is sensitive to the solution concept of the bargaining game. Indeed, let us analyze the same problem under the $R-$solution.

In case (i), in the bilateral negotiations players S and B equally split the value they can guarantee for themselves, x. Hence S's disagreement point in the trilateral negotiation is $T_S = \frac{x}{2}$. Since the value added by the grand coalition is y then player S's payoff is given by

$$U_s = \frac{x}{2} + \frac{y}{3}.$$

Therefore, in this case player S chooses $\overline{x}$. Since there is only one two-player coalition with a strictly positive value, player S's payoff is identical to the one obtained under the Shapley value.

In case (ii), player S has nothing to bargain about exclusively with player E (since player B is indispensable) or with player B (since player E owns the asset). Therefore, his disagreement point in the trilateral negotiation is

$T_s = 0$. Since the value added by the grand coalition is x player S's payoff is

$$U_s = \frac{x}{3}.$$

Therefore, in this case player S chooses $\underline{x}$. Once again, player S's payoff is identical to the one obtained under the Shapley value.

Finally, in case (iii) there are two two-player coalitions, (B, S) and (B, E), and then with a strictly positive value the outcome of simultaneous, bilateral negotiations depend on the difference between x and y:

If $\frac{x}{2} > y$, player S is able to capture one half of his contribution to his coalition with B. Thus, in this region $T_S = \frac{x}{2}$. Since the grand coalition adds a surplus of y, his final payoff is $U_s = \frac{x}{2} + \frac{y}{3}$, which is identical to the Shapley value.

If $x > y \geq \frac{x}{2}$,in the bilateral negotiations player S can guarantee to himself his entire "competitive advantage" with respect to player E vis-a-vis player B. That is, his disagreement point in the trilateral negotiation is $T_S = x - y$. Hence, his payoff is $U_s = x - \frac{2y}{3}$.

If $x \leq y$, player S is in an extremely weak position in the bilateral negotiations, so that he cannot guarantee anything positive for himself. Thus, the disagreement point in the trilateral negotiation is $T_S = 0$. Since the value added by the grand coalition is x, then $U_s = \frac{x}{3}$.

Therefore, player S's ex-ante expected utility is given by

$$\begin{aligned} EU_s &= H\left(\frac{x}{2}\right)\frac{x}{2} + \left[H(x) - H\left(\frac{x}{2}\right)\right]x + [1 - H(x)]\frac{x}{3} + \\ &\quad \frac{1}{3}\int_0^x ydH(y) - \int_{\frac{x}{2}}^x ydH(y), \end{aligned}$$

and then his choice of investment, $\widehat{x}$, satisfies

$$\Psi'(\widehat{x}) = \frac{1 + 2H(\widehat{x})}{3} - \frac{1}{2}H(\frac{\widehat{x}}{2}).$$

This may be higher or lower than the level predicted under the Shapley value, $\overline{x}$. If investment costs are sufficiently low so that $H(\widehat{x})$ approaches

1, then the incentives to invest are enhanced if the tycoon owns the yacht. However, if costs are sufficiently high, then $H(\widehat{x})$ will be closer to 0 and investment is discouraged if the tycoon owns the yacht. In other words, under the $R-$solution the model will not unambiguously predict which is the most efficient ownership structure unless we have more information about the relative competitive position of S and E.

Conclusion 2 *Under the $R-$solution it may not be optimal that player B (the indispensable player) owns the asset.*

4.2 Exclusive contracts (Segal and Whinston, 2000)

Segal and Whinston (2000), SeW, study the role of exclusive contracts in the protection of relation-specific investments. Their main insight is that, under complete information, exclusive contracts are irrelevant in protecting the incumbent seller's relation-specific investment, unless such investment has an externality on the entrant. In fact, this is a somewhat counterintuive result that contradicts the received wisdom (for instance, Klein 1988, Marvel 1982, or Masten and Sneyder 1993).

In this subsection we examine the robustness of this result presented in their Section 2. Let player B (buyer) derive a potential utility $w = 1$ (independent of x and y) from one unit of the consumption good taht can be provided by both, S and E at a cost that for simplicity we assume $c_s(x) = 1-x$ and $c_e(y) = 1-y$ respectively. No particular asset is necessary to produce the good or, equivalently, both S and E have access to the necessary asset. In period 0 players S and B may or may not sign an exclusivity contract. If they do, then in period 2 player B cannot purchase from E without S's permission.

The first-best level of investment, x^*, is the solution to the problem of choosing x in order to minimize total costs:

$$\int_0^x (1-x)\,dH(y) + \int_x^1 (1-y)\,dH(y) + \Psi(x).$$

Thus,

$$H(x^*) = \Psi'(x^*).$$

As a solution concept for the renegotiation at period 2, SeW consider the case where in period 2 the entrant is willing to supply the good at a price $p_e = 1 - y$. That is, the entrant has no bargaining power, perhaps as a consequence of there being many identical entrants with the same (uncertain at time 1) cost. Players B and S split equally any surplus from renegotiation over the disagreement point, which depends on whether or not they had previously signed an exclusive contract.

In the absence of any contract, if $x < y$, B always prefers to purchase from E, and player S makes zero profits: $U_s = 0$. If $x \geq y$ then it is efficient that B purchases from S. However, when they bargain over the price, player S has no outside option, $t_S = 0$, but player B can threaten to switch and purchase from E at a price equal to $1 - y$ and make profits $t_B = y$. Since they split the surplus over the disagreement point, $x - y$, then $U_s = \frac{x-y}{2}$. Thus, under non-exclusivity, in period 1 player S's expected payoff (gross of investment cost) is

$$EU_s^{ne} = \frac{1}{2}\left[H(x)x - \int_0^x y dH(y)\right].$$

Since player S chooses x in order to maximize $EU_s^{ne} - \Psi(x)$, then the solution, $\overline{x}$, is given by the first order condition:

$$\frac{1}{2}H(\overline{x}) = \Psi'(\overline{x}).$$

In this case, as expected, there is underinvestment: $\overline{x} < x^*$. The entrant obtains no surplus but whenever player S is the most efficient supplier he shares equally the returns from his investment with player B.

Under an exclusive contract, the relative bargaining position of players B and S in period 2 change dramatically. If $x \geq y$ then it is efficient that B purchases from S. They split the surplus equally but now the disagreement points are also 0 for both players: B can no longer threaten with purchasing from E without B's permission. Hence, $U_s = \frac{x}{2}$. If $x < y$ the efficient trade (B purchases from E at a price $1-y$) requires an agreement between B and S. They split a surplus of y and the disagreement points are once again 0 for both players. Hence, $U_s = \frac{y}{2}$. Thus, under exclusivity, player S's expected utility in period 1 (gross of investment cost) is:

$$EU_s^e = \frac{1}{2}\left[H(x)\,x + \int_x^1 ydH(y)\right].$$

Therefore, under exclusivity player S chooses the same investment level, $\overline{x}$, that under non-exclusivity. Indeed, note that

$$EU_s^e - EU_s^{ne} = \frac{1}{2}\int_0^1 ydH(y).$$

That is, an exclusivity contract is an effective instrument to capture additional surplus (at the expense of E), but this additional surplus does not depend on the investment x, and then it does not help protecting relation-specific investments.

Conclusion 3 *Under a competitive entrant and Nash bargaining between the buyer and the incumbent seller, an exclusivity contract is irrelevant from an efficiency point of view. With or without exclusivity the level of investment is inefficiently low.*

In their general model, SeW assume a more general bargaining model in period 2, with the only restriction that payoffs are linear to marginal contribution of players. That is, a generalization of the Shapley value. The main result holds. Let us see that this is indeed the case. In the case of nonexclusivity, we have:

$$V = \max\{x, y\}, \tag{6}$$

$$v_{SB} = x, \tag{7}$$

$$v_{BE} = y. \tag{8}$$

The rest of coalitions have a value of 0. Under the exclusivity contract, equation (8) is replaced by

$$v_{BE} = 0. \tag{9}$$

Now consider the incentives to invest assuming Shapley value payoffs with no exclusivity. S's payoff equals $U_s^{ne} = \frac{1}{3}\max\{x - y, 0\} + \frac{x}{6}$. Thus, S's expected utility in period 1 is

$$EU_s^{ne} = H(x)\frac{x}{2} + [1 - H(x)]\frac{x}{6} - \frac{1}{3}\int_0^x ydH(y).$$

Therefore, the first order condition for the maximization of $EU_s^{ne} - \Psi(x)$ is

$$\frac{1}{6}(2H(x) + 1) - \Psi'(x) = 0. \tag{10}$$

Note that in this case, player S's optimal value of x may be higher or lower than x^*. Indeed, evaluated at x^*, the left hand side above equals $\frac{1}{6} - \frac{2}{3}H(x^*)$. Therefore, if investment is very costly for S, so that $H(x^*)$ is small, then there is overinvestment, and otherwise there will be underinvestment again.

On the other hand, under exclusivity, player S's payoff is equal to $U_s^e = \frac{1}{3}\max\{x, y\} + \frac{x}{6}$. Thus, S's expected utility in period 1 is

$$EU_s^e = H(x)\frac{x}{2} + [1 - H(x)]\frac{x}{6} + \frac{1}{3}\int_x^1 ydH(y).$$

Once again, note that the difference $EU_s^e - EU_s^{ne}$ is constant in x:

$$EU_s^e - EU_s^{ne} = \frac{1}{3}\int_0^1 ydH(y).$$

Conclusion 4 *Under the Shapley value, an exclusivity contract is irrelevant from an efficiency point of view. In either case, it is ambiguous whether or not there is underinvestment.*

Let us now analyze the same problem when we use the $R-$solution to predict payoffs in period 2. In that case, player S's payoff is

$$U_s^{ne} = \begin{cases} \frac{x}{2}, \text{ if } y \leq \frac{x}{2} \\ x - y, \text{ if } x \geq y \geq \frac{x}{2} \\ 0, \text{ if } x < y \end{cases}$$

Thus, its expected payoff in period 1 is given by

$$EU_s^{ne} = H\left(\frac{x}{2}\right)\frac{x}{2} + \left[H\left(x\right) - H\left(\frac{x}{2}\right)\right]x - \int_{\frac{x}{2}}^{x} ydH\left(y\right).$$

Note that under non-exclusivity investment incentives are inefficiently low. The question is whether or not exclusivity may help reducing the underinvestment problem.

Under exclusivity, if $y \leq x$ then players B and S split their surplus, x, without any interference from E. Hence, $U_s^e = \frac{x}{2}$. If $y \geq x$ then player x arrives at the grand coalition securing $\frac{x}{2}$. But now, the added value of the grand coalition is $y - x$. As a result, player S's payoff is $U_s^e = \frac{x}{6} + \frac{y}{3}$. Thus, its expected payoff in period 1 is given by

$$EU_s^e = H\left(x\right)\frac{x}{2} + \left[1 - H\left(x\right)\right]\frac{x}{6} + \frac{1}{3}\int_x^1 ydH\left(x\right).$$

Therefore, the difference $EU_s^e - EU_s^{ne}$ is not independent of x:

$$-\left[H\left(x\right) - H\left(\frac{x}{2}\right)\right]\frac{x}{2} + \frac{1}{3}\int_x^1 ydH\left(x\right) + \int_{\frac{x}{2}}^{x} ydH\left(y\right).$$

Exclusivity undermines player S's ability to capture the return on his investment efforts if $\frac{x}{2} \leq y \leq x$, but in contrast it enhances his ability whenever $y > x$. Therefore, if investment costs are sufficiently low, then $H(x)$ is close to 1 and exclusivity actually hurts investment incentives. In this case, the paradoxical result obtained by SeW is exacerbated. However, if investment costs are sufficiently high, then $H(x)$ will be close to 0, and exclusivity improves investment incentives. In other words, under the $R-$solution exclusivity helps protecting relation-specific investment only when the seller's competitive position is sufficiently weak. Exclusivity is useful only when there is a lot to protect. Finally, the first order conditions for maximization of s's profits under exclusivity coincide under the Shapley value and the $R-$solution, (10). Thus, if investment costs are sufficiently high (x low), then the level of investment under exclusivity will be inefficiently high.

Conclusion 5 *Under the $R-$solution an exclusivity contract may increase or decrease S's incentives to invest. If investment costs are sufficiently high then an exclusivity contract reduces the underinvestment problem. If costs are even higher then exclusivity may overprotect his relation-specific investment, in the sense that the investment level may be inefficiently high.*

4.3 Stipulated damages and entry barriers (Spier and Whinston, 1995)

Spier and Whinston (1995), SpW, study how contracts between a buyer and an incumbent seller may prevent entry by an alternative seller. The type of contracts they consider have more structure than in those considered by SeW. In particular, they specify a price, p, and the level of damages, b, that B has to pay S in case she breaches the contract (and purchases from E). In fact, it can be shown that, for all renegotiation concepts considered below, it is optimal to stipulate an arbitrarily large b (which is equivalent to an exclusive contract), since it can always be renegotiated down whenever a trade between B and E is efficient.[5] Thus, players S and B can always guarantee for themselves $p-(1-x)$ and $1-p$, respectively.

SpW, assumes that the entrant has a first-mover advantage, while the incumbent and the buyer engage in bilateral bargaining. At the beginning of period 2 the entrant makes an irreversible price offer, p_e, and next players B and S equally split any surplus exceeding their disagreement point.

[5] In order to prove that the optimal contract includes an arbitrarily high level of damages, we need to consider the performance of contracts with relatively low values of b. In this case, the $R-$solution to simultaneous, bilateral negotiations includes a positive probability to inefficient trades (Region 3 in Section 2).

The entrant's optimal strategy is to set $p_e = 1 - x$ whenever he is the most efficient supplier ($y \geq x$). On the contrary, if the incumbent seller is more efficient ($y < x$) then any price above his own cost, $p_e \geq 1 - y$, will be unable to attract the buyer. Thus, if $y < x$ then S and B have nothing to renegotiate, since the sum of their threat points is equal to the value of their trade. Moreover, nothing can be gained by letting B purchase from E at a price $p_e \geq 1 - y > 1 - x$. Thus, the payoff of players S and B are given respectively by $U_s = p - (1 - x)$, $U_b = 1 - p$. On the other hand, if $y \geq x$ then the coalition between B and S is indifferent between trading at the preestablished price p and letting B purchase from E at the price $p_e = 1 - x$. Player S will agree to implement the second option only if the renegotiated level of damages, b', satisfy $b' \geq p - (1 - x)$. Similarly, player B will agree if and only if $1 - (1 - x) - b' \geq 1 - p$. These two conditions can only hold if $b' = p + x - 1$. Hence, $U_s = p - (1 - x)$ and $U_b = 1 - p$.

Summarizing, in the case where the entrant enjoys a first-mover advantage then player S's expected utility in period 1 is

$$EU_s = x + p - 1.$$

Also note that B's expected utility does not depend on x. Therefore, the level of investment chosen by S is also the level which the coalition $\{B, S\}$ finds optimal. In particular, player S chooses $\widetilde{x}$, which is given by

$$1 = \Psi' (\widetilde{x}) .$$

Clearly, $\widetilde{x} > x^*$.

Conclusion 6 *If the entrant enjoys a first-mover advantage and buyer and seller engage in bilateral renegotiation, then the optimal contract generates overinvestment, which in this context implies that investment is an effective entry barrier, since it reduces the frequency of transactions realized by the entrant below the efficient level.*

Let us now analyze the same problem using the $R-$solution to predict payoffs in period 2. (In this particular example, the predictions of the Shapley value coincide with those of the $R-$solution, conditional on the optimal contract.)

If $y < x$ then the value of all coalitions are the following:

$$V = x,$$

$$v_{BS} = x,\ v_E = 0,$$

$$v_{BE} = 1 - p,\ v_S = p - (1 - x),$$

$$v_{SE} = p - (1 - x)\,,\ v_B = 1 - p.$$

Note that in this case none of the three two-player coalitions provide any positive net value, i.e., for all $i, j = B, S, E$, $v_{ij} = v_i + v_j$. Moreover, the grand coalition provides no additional value: $V = v_{BS}$. Therefore, the final payoffs coincide with v_i, for $i = B, S, E$. In particular, $U_s = x + p - 1$, which is identical to the result obtained in SpW.

If $y \geq x$ then the value of all coalitions are the following:[6]

$$V = y,$$

$$v_{BS} = x,\ v_E = 0,$$

$$v_{BE} = 1 - p,\ v_S = p - (1 - x),$$

$$v_{SE} = p - (1 - y)\,,\ v_B = 1 - p.$$

In this case, there is only one coalition that provides a positive net value, $v_{SE} > v_S + v_E$. Thus,

$$T_s = \frac{x + y}{2} - (1 - p)\,,$$

$$T_b = 1 - p,$$

[6] The way we have written v_{SE} presumes that player S can deliver the good to B (and fullfill his contractual obligations), which is actually produced by E. If such a transaction were not feasible, and hence $v_{SE} = v_B = 0$, the qualitative results would not change.

$$T_e = \frac{y - x}{2}.$$

Since these three fall-back options add up to y, then the final payoffs coincide with these values. Therefore, player S's expected utility is

$$EU_s = H(x)\,x + [1 - H(x)]\,\frac{x}{2} + \frac{1}{2}\int_x^1 y dH(y) - (1 - p)\,.$$

Thus, player S chooses $x = \widehat{x}$, $x^* < \widehat{x} < \widetilde{x}$. In this case the application of the $R-$solution does not change the qualitative results of the original paper, although it does moderate the strength of their results.

Conclusion 7 *Under the $R-$solution there is also overinvestment, but the size of the inefficiency is lower than in the case of an entrant with a first-mover advantage.*

4.4 Discussion

In this section we have applied the $R-$solution to various existing economic models. In the original papers, authors use either ad hoc restrictions on the structure of the renegotiation game and/or the Shapley value. The variety of approaches reflects the lack of consensus on how to approach these three-player bargaining games. In particular, some authors reveal their preference to make special assumptions about the structure of the game rather than use a general solution concept like the Shapley value.

In the discussion of these examples, we have shown that some important qualitative results are not robust to changes in the solution concept. Therefore, the modeling choices over the renegotiation game are, in fact, crucial. Papers like HM and SeW contained surprising results that tended to contradict to some extent the received wisdom. If we apply the $R-$solution to these models it turns out that important qualitative results become ambiguous. In other words, these models seem to be able to accommodate the traditional intuitions as well as the counterintuitive results they emphasized. For instance, under the $R-$solution, if the incumbent seller's competitive position with respect to potential entrants is relatively weak, then an exclusivity contract will help protecting their relation-specific investments, as it had been frequently argued in the past. However, if the incumbent's competitive position is relatively strong, then an exclusivity contract may actually reduce the sensitivity of the incumbent's payoff to their investment and hence exacerbate the underinvestment problem, as suggested by SeW.

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6 Appendix

6.1 Proof of Proposition 1:

First we propose an $\epsilon - R$ two-player solution for the game (N, v) for ϵ small enough. This will show existence. To save in notation, ,we will dispose of the (ϵ) index of the solution.

1) If $v_{13} < \frac{1}{2}v_{12}$ (so that $v_{12} \geq v_{13} + v_{23}$ is also satisfied), consider $u_1^{12} = u_1^{12} = \frac{1}{2}v_{12}$, and $u_i^{ij} = 0$ for all $i, j \neq 1, 2$. Also, let $p_{12} = 1 - \epsilon$ and any p_{13}, p_{23} so that $p_{13} + p_{23} = \epsilon$. For this to be a solution, we need $t_3^{i3} = 0$ for $i = 1, 2$. Also, in this case

$$t_i^{i3} = \frac{(1-\epsilon)\frac{1}{2}v_{12}}{1 - p_{i3}},$$

and also $t_1^{12} = t_2^{12} = 0$. Note that $t_i^{i3} \geq (1-\epsilon)\frac{1}{2}v_{12} > (1-\epsilon)v_{i3}$ since $\frac{1}{2}v_{12} > v_{i3}$. Thus, for ϵ small enough $t_i^{i3} > v_{i3}$. Thus all condititons are satified and therefore this is an $\epsilon - R$ two-player solution exists for the game (N, v).

2) If $v_{13} = \frac{1}{2}v_{12} > v_{23}$, consider $u_1^{12} = u_1^{12} = \frac{1}{2}v_{12}$ $(= v_{13})$, and $u_2^{23} = u_3^{23} = 0$. Also, let $p_{12} = 1 - \epsilon$ and $p_{13} = 0$, $p_{23} = \epsilon$. Then $t_1^{12} = 0$. Also, $t_2^{23} = \frac{(1-\epsilon)\frac{1}{2}v_{12}}{(1-\epsilon)} = \frac{1}{2}v_{12} > v_{23}$. Therefore, $u_2^{23} = 0$ and consequently $t_2^{12} = 0$. Thus this is indeed the solution if complete it with $t_1^{13} = (1-\epsilon)\frac{1}{2}v_{12}$, $t_1^{13} = 0$, $u_1^{13} = (1 - \frac{\epsilon}{2})v_{13}$ and $u_3^{13} = \frac{\epsilon}{2}$.

3) If $v_{13} = v_{23} = \frac{1}{2}v_{12}$, consider $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$ $(= v_{i3},\ i = 1, 2)$, $p_{12} = 1 - \epsilon$ and $p_{13} = p_{23} = \frac{\epsilon}{2}$. Then $t_1^{13} = t_2^{23} = \frac{(1-\epsilon)\frac{1}{2}v_{12}}{(1-\frac{\epsilon}{2})} < v_{i3}$, $i = 1, 2$. Also, consider $u_3^{13} = u_3^{23} = A > 0$. Thus, t_3^{i3}, $i = 1, 2$, will have to satisfy:

$$t_3^{i3} = \frac{\frac{\epsilon}{2}A}{1 - \frac{\epsilon}{2}}, \text{ and}$$

$$A = \frac{1}{2}\left(v_{i3} - \frac{(1-\epsilon)\frac{1}{2}v_{12}}{1 - \frac{\epsilon}{2}} + \frac{\frac{\epsilon}{2}A}{1 - \frac{\epsilon}{2}}\right),$$

and solving for A taking into account that $\frac{1}{2}v_{12} = v_{i3}$, we obtain

$$A = \frac{\epsilon v_{i3}}{4 - 3\epsilon}.$$

Note that for ϵ small $t_3^{i3} + t_i^{i3}\ v_{i3}$, $i = 1, 2$. Also, note that given these values for U_i^{i3}, we have $t_1^{12} = t_2^{12} = \frac{\frac{\epsilon}{2}(v_{i3}-A)}{\epsilon} = \frac{(v_{12}-2A)}{4}$, so that $t_1^{12} + t_2^{12} < v_{12}$. Thus, $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$ satisfies the definition.

4) If $v_{12} \geq v_{13} + v_{23}$ but $v_{13} > \frac{1}{2}v_{12}$, then consider $u_1^{12} = u_1^{13} = u_1$, to be obtained later, with $0 < u_1 < v_{13}$, and $u_2^{23} = u_3^{23} = 0$. Note that this implies that $u_2^{12} = v_{12} - u_1 > u_2^{23}$ and $u_3^{13} = v_{13} - u_1 > u_3^{23}$, which implies that $p_{23} \leq \epsilon$. Let $p_{23} = \epsilon$. Then $p_{12} = 1 - \epsilon - p_{13}$. Finally, $u_2^{23} = u_3^{23} = 0$ implies that $t_2^{12} = t_3^{13} = 0$, and we can then check that $t_2^{12} + t_2^{12} < v_{12}$, whereas

$$\begin{aligned} t_1^{13} + t_3^{13} &= \frac{p_{12}u_2^{12}}{1-\epsilon} + \frac{p_{13}u_3^{13}}{1-\epsilon} \\ &= (v_{12} - u_1) - \frac{p_{13}(v_{12} - v_{13})}{1-\epsilon}. \end{aligned}$$

We will propose u_1 sufficiently close to v_{13} so that $t_1^{13} + t_3^{13} \leq v_{13}$. In that case, the definition of solution in u_1^{12} and u_1^{13} gives us the following two equations

$$u_1 = \frac{1}{2}(v_{13} + \frac{(1-\epsilon-p_{13})u_1}{1-p_{13}}) = \frac{1}{2}(v_{12} + \frac{p_{13}u_1}{1-p_{13}}).$$

This is a system with two unknowns. Note that if we have a (valid) solution to this system, then as ϵ approaches 0 the first equation approaches $u_1 = \frac{1}{2}(v_{13}+u_1)$ whose only solution is $v_{13} = u_1$. (For positive ϵ, indeed $u_1 < v_{13}$.) Thus, for ϵ small enough, $t_1^{13} + t_3^{13} < v_{12} - u_1 = v_{12} - 2v_{13} + v_{13} + (v_{13} - u_1)$ and the right hand side converges to $v_{12} - 2v_{13} + v_{13} < v_{13}$. Also, solving for u_1, we can write the system as

$$v_{13}\left(2 - \frac{p_{13}}{p_{13}+\epsilon}\right) = 2 - \frac{p_{13}+\epsilon}{p_{13}}.$$

This is a quadratic equation in p_{13} with only a positive root that converges to 0 as ϵ converges to zero. Thus, the proposed solution satisfies the conditions for ϵ small enough. And for ϵ small, p_{12} is close to 1.

5) If $v_{12} < v_{13} + v_{23}$, then propose $u_i^{ij} = u_i^{ik} = u_i > 0$, for all $i, j, k = 1, 2, 3$, $i \neq j \neq k, i \neq k$. Then the definition of u_i^{ij} requires that $u_i + u_j = v_{ij}$ for all ij. This is a system of three linear (independent) equations with solution $u_i = \frac{v_{ij}+v_{ik}-v_{jk}}{2}$. Also, $t_i^{ij} = \frac{p_{ik}U_i}{1-p_{ij}}$. Finally, p is the solution to

$$\begin{aligned} u_1 &= \frac{1}{2}(v_{12} + \frac{p_{13}U_1}{p_{13}+p_{23}} - \frac{p_{23}U_2}{p_{13}+p_{23}}) \\ u_2 &= \frac{1}{2}(v_{23} + \frac{p_{12}U_2}{p_{13}+p_{12}} - \frac{p_{13}U_3}{p_{13}+p_{12}}) \\ u_3 &= \frac{1}{2}(v_{13} + \frac{p_{12}U_1}{p_{23}+p_{12}} - \frac{p_{23}U_3}{p_{23}+p_{12}}). \end{aligned}$$

(Note that if these $u_i + u_j = v_{ij}$ and $u_i = \frac{1}{2}(v_{12}+ t_i^{ij}- t_j^{ij})$ implies that $u_j = \frac{1}{2}(v_{12}+ t_j^{ij}- t_i^{ij})$). Taking into account $u_i + u_j = v_{ij}$, these equations can be written as

$$\begin{aligned} -p_{13}u_2 + p_{23}u_1 &= 0 \\ -p_{12}u_3 + p_{13}u_2 &= 0 \\ -p_{12}u_3 + p_{23}u_1 &= 0. \end{aligned}$$

Note that the third equation is simply the sum of the previous two. That is, there are only two linearly independent equations. Thus, two of these equations plus $p_{13} + p_{23} + p_{23} = 1$ form a linear system with a unique solution. The solution is a probability distribution, since all three variables take positive values. Indeed, the first two equations cn be written as $\frac{p_{13}}{u_1} = \frac{p_{23}}{u_2}$ and $\frac{p_{12}}{u_2} = \frac{p_{13}}{u_3}$, so that all solution vectors to these two equations have either all positive components or all negative. And no solution with all negative components satisfies the equation $p_{13}+p_{23}+p_{23} = 1$. Finally, note that $t_j^{ij}+ t_i^{ij} = \frac{p_{jk}u_j}{p_{jk}+p_{ik}} - \frac{p_{ik}u_i}{p_{jk}+p_{ik}}$, so that since both $u_j, u_i < v_{ij}$, indeed $t_j^{ij}+ t_i^{ij} < v_{ij}$.

Next, we can simply check that if we select the $\epsilon - R$ SSBN that we have just characterized, then the $\lim_{\epsilon\to 0} \{U(\epsilon), t(\epsilon), p(\epsilon)\}$ is as stated in the proposition. Thus, we only need showing that there is no other triple $\{u, t, p\}$ that is the limit of a sequence of $\epsilon - R$ SSBN as ϵ approaches 0. We do so by contradiction.

a) If $v_{13} \leq \frac{1}{2}v_{12}$ (so that $v_{12} \geq v_{13} + v_{23}$ is also satisfied), consider a limit where u_1^{12},$u_1^{12} \neq \frac{1}{2}v_{12}$. Note that this implies that either $u_i^{12} > u_j^{12}$ for some $i = 1, 2$, $i \neq j$, or $u_1^{12} = u_2^{12} = 0$. In the latter case, this means that for ϵ small $t_1^{12}+ t_2^{12} > v_{12}$. But for any ϵ, $t_1^{12} \leq v_{13}$ and $t_2^{12} \leq v_{23}$. Thus, since $v_{12} \geq v_{13} + v_{23}$ we obtain a contradiction. Now, if $u_i^{12} > u_j^{12}$ for some $i = 1, 2$, $i \neq j$, then the limit $p_{12} < 1$. Indeed, for the limit $u_i^{12} > \frac{1}{2}v_{12}$ we need to have in the limit $t_i^{12} > 0$. Thus, $u_i^{i3} > 0$ for ϵ small, which is a contradiction if p_{12} is close to 1, since

$$t_i^{i3} = \frac{p_{12}}{1 - p_{i3}} u_i^{12} > \frac{p_{12}}{1 - p_{i3}} \frac{1}{2} v_{12},$$

and the right hand side is larger than $v_{i3} \leq \frac{1}{2}v_{12}$ if p_{12} is close enough to 1.

Thus we can only have the limit $u_1^{12} \neq u_2^{12}$ if for ϵ small p_{i3} is larger than ϵ for $i = 1$ or 2. This in turn means that for ϵ small (and then in the limit as $\epsilon \to 0$), $u_i^{i3} \geq u_i^{12}$ and $u_3^{i3} \geq u_3^{j3}$ for some $i = 1$ or 2 and $j \neq i$. Assume $i = 1$. That means that in the limit $u_1^{12} \leq u_1^{13} \leq v_{13} \leq \frac{1}{2}v_{12}$. Thus, w can only hve that $u_2^{12} > \frac{1}{2}v_{12}$. Again, that means that $t_2^{12} = \frac{p_{23}u_2^{23}}{1-p_{12}}$ does not approach zero as the ϵ approaches 0. Therefore, again $p_{23} > \epsilon$ for ϵ small, which implies that $u_2^{23} \geq u_2^{12}$ for ϵ small, and this is a contradiction, since in the limit $u_2^{12} > v_{23}$. We can use exactly the same argument to exclude the alternative case, where $i = 2$. Thus, we conclude that in this region the limit of any sequence of $\epsilon - R$ SSBN when $\epsilon \to 0$ satisfies $u_1^{12} = u_1^{12} = \frac{1}{2}$. It follows that $p_{12} = 1$. Indeed, if $p_{13} > 0$ in the limit, then for ϵ small $u_1^{13} \geq u_1^{12} = v_{13}$. This is only possible if $t_1^{13} = \frac{p_{12}}{1-p_{13}} u_1^{12} = v_{13}$, which requires $p_{12} = 1 - p_{13}$ since $u_1^{12} = v_{13}$. Thus, this requires that $p_{23} = 0$. In this case, $t_2^{12} = 0$, whereas $t_1^{12} = \frac{p_{13}}{1-p_{12}} u_1^{13} = v_{13}$, and then $u_1^{12} > u_2^{12}$, which contradicts $u_1^{12} = u_2^{12}$. The same argument shows that $p_{23} > 0$ is a contradiction.

b) If $v_{12} \geq v_{13} + v_{23}$ but $v_{13} > \frac{1}{2} v_{12}$ and $v_{13} < v_{12}$, as in the previous case, $u_1^{12} = u_2^{12} = 0$ implies that for ϵ small $t_1^{12} + t_2^{12} > v_{12}$. But for any ϵ, $t_1^{12} \leq v_{13}$ and $t_2^{12} \leq v_{23}$. Thus, again, since $v_{12} \geq v_{13} + v_{23}$ we obtain a contradiction. We can also reject the possibility that both $p_{13} > 0$ and $p_{23} > 0$. Indeed, that would imply that for ϵ small $u_1^{13} \geq u_1^{12}$ and $u_2^{23} \geq u_2^{12}$ whereas $u_1^{13} \leq v_{13}$ and $u_2^{23} \leq v_{23}$, so that this would imply $u_1^{12} + u_2^{12} = v_{12} \geq v_{13} + v_{23}$. This is only possible if for ϵ small $v_{12} = v_{13} + v_{23}$, $u_1^{13} = u_1^{12} = v_{13}$, and $u_2^{23} = u_2^{12} = v_{23}$, so that $u_3^{13} = u_3^{23} = 0$. But then for $i = 1, 2$ we would have $t_i^{13} = \frac{p_{12}}{1-p_{i3}} v_{i3} < v_{i3}$, which contradicts $u_3^{i3} = 0$. Thus, we have that $p_{i3} = 0$ for $i = 1$ or 2. Assume now that $p_{13} = 0$ but $p_{23} > 0$ in the limit. Thus, $t_1^{12} = \frac{p_{13}}{p_{13}+p_{23}} u_1^{13} \to 0$, and $t_2^{12} = \frac{p_{23}}{p_{13}+p_{23}} u_2^{23} \geq \frac{p_{23}}{p_{13}+p_{23}} u_2^{12}$ for ϵ small, where this last inequality follows from the fact that for ϵ small $p_{23} > \epsilon$. Thus, from the definition of u_i^{ij}, we have that for ϵ small $u_2^{12} = \frac{1}{2}\left(v_{12} + t_2^{12} - t_1^{12}\right) \geq \frac{1}{2}\left(v_{12} + \frac{p_{23}}{p_{13}+p_{23}} u_2^{12} - t_1^{12}\right) \to \frac{1}{2}\left(v_{12} + u_2^{12}\right)$. Thus, in the limit $u_2^{12} = v_{12} > v_{23}$, and this contradicts that $u_2^{23} \geq u_2^{12}$. The same argument precludes that $p_{13} > 0$ and $p_{23} = 0$ in the limit unless $v_{12} = v_{13}$.

Thus, in the limit when $v_{13} < v_{12}$ we must have $p_{12} = 1$. Also, this implies that in the limit $t_2^{23} = u_2^{12}$ and $t_1^{13} = u_1^{12}$, whereas $t_3^{13} = t_3^{23} = 0$. If $v_{12} > v_{13} + v_{23}$, and since $t_2^{23} + t_1^{13} = v_{12}$, we must have that for ϵ small either $t_1^{13} + t_3^{13} > v_{13}$ or $t_2^{23} + t_3^{23} > v_{23}$, or both, and therefore either $u_1^{13} = 0$ or $u_2^{23} = 0$ or both. In latter case we have that for $\epsilon > 0$ the only solution would be $u_2^{12} = u_1^{12} = \frac{1}{2} v_{12} < v_{13}$ which is a contradiction with $t_1^{13} + t_3^{13} > v_{13}$. Also, if for ϵ small $u_1^{13} = 0$ and $u_2^{23} > 0$ then $u_1^{12} \leq \frac{1}{2} v_{12} < v_{13}$, which again contradicts $t_1^{13} + t_3^{13} > v_{13}$. Thus, we only have to consider the case that for ϵ small $u_1^{13} > 0$ and $u_2^{23} = 0$ so that $t_2^{12} = 0$. Note that $u_1^{12} \geq u_1^{13}$ since $p_{12} > \epsilon$ for ϵ small. Also, if $u_1^{12} > u_1^{13}$ in the limit, then for ϵ small $t_1^{13} = \frac{p_{12}}{p_{12}+p_{23}} u_1^{12} > u_1^{13}$, which could only happen if $u_1^{13} = 0$, which would be a contradiction. Thus, we have that for ϵ small $u_1^{12} = u_1^{13}$. This is exactly the case we have considered in 4) above, so that the result will be exactly the one obtained there.

c) Consider the case $v_{12} = v_{13}$ and then $v_{12} = v_{13} + v_{23}$ with $v_{23} = 0$, and show that we can only have $u_1^{12} = u_1^{13} = v_{12}$, and therefore $u_i^{ij} = 0$ for $i \neq 1$, $j \neq i$. Note that for any ϵ, $u_i^{23} = 0$ and so $t_i^{1i} = 0$ for $i = 2, 3$. This guarantees that $t_1^{1i} + t_i^{1i} \leq v_{1i}$ for $i = 2, 3$. Now, assume that in the limit

$v_{12} > u_1^{12}, u_1^{13}$. This means that $u_2^{12}, u_3^{13} > 0$. Then for ϵ small, $p_{23} \leq \epsilon$.

Assume moreover that in the limit $u_1^{12} > u_1^{13}$. Then also for ϵ small $p_{13} \leq \epsilon$. In this case $1 - \epsilon \geq p_{12} \geq 1 - 2\epsilon$. Thus $t_1^{13} = \frac{p_{12}}{1-p_{13}} u_1^{12} \geq \frac{1-2\epsilon}{1} u_1^{12}$ and $t_1^{12} = \frac{p_{13}}{1-p_{12}} u_1^{13} < u_1^{13}$. Thus, for ϵ small we get a contradiction: $t_1^{13} > t_1^{12}$, which implies that $u_1^{13} \geq u_1^{12}$ in the limit. We get the same contradiction if we assume that $u_1^{12} < u_1^{13}$. Now, assume that in the limit $u_1^{12} = u_1^{13} = u_1$. Given ϵ we must have

$$
\begin{aligned}
u_1^{12} &= \frac{1}{2}(v_{12} + \frac{p_{13}}{p_{13} + p_{23}} u_1^{13}) \\
u_1^{13} &= \frac{1}{2}(v_{12} + \frac{p_{12}}{p_{12} + p_{23}} u_1^{12}).
\end{aligned}
$$

Since u_1^{12} converges to the same value as u_1^{13} then for $\frac{p_{13}}{p_{13}+p_{23}}$ and $\frac{p_{12}}{p_{12}+p_{23}}$ converge to the same value which has to be 1, since p_{23} converges to 0. Then we conclude that $u_1 = v_{12}$.

d) Assume $v_{12} < v_{13} + v_{23}$. We will show that in the limit we must have $u_i^{ij} = u_i^{ik}$ for all $i, j, k = 1, 2, 3$, $i \neq j \neq k$, $i \neq k$.

d.1) First, assume that in the limit $u_1^{12} > u_1^{13}$ and $u_2^{12} > u_2^{23}$. Then for ϵ small we must have $p_{13}, p_{23} \leq \epsilon$. Also, since $u_1^{12} > 0$ and $u_2^{12} > 0$, we must have that for ϵ small $t_1^{12} + t_2^{12} \leq v_{12}$ and so $u_1^{12} + u_2^{12} = v_{12}$. This will also happen in the limit. Then, in the limit either $u_1^{12} < v_{13}$ or $u_2^{12} < v_{23}$ since $v_{12} < v_{13} + v_{23}$. If $u_1^{12} < v_{13}$ then

$$
t_1^{13} + t_3^{13} = \frac{p_{12} u_1^{12}}{1 - p_{13}} + \frac{p_{23} u_3^{23}}{1 - p_{13}} < u_1^{12} + \epsilon v_{23}.
$$

The right hand side converges to $u_1^{12} < v_{13}$. That is, for ϵ small $t_1^{13} + t_3^{13} < v_{13}$. Thus, for ϵ small

$$
u_1^{13} = \frac{1}{2}\left(v_{13} + t_1^{13} - t_3^{13}\right) \geq \frac{1}{2}\left(v_{13} + (1 - 2\epsilon) u_1^{12} - \frac{\epsilon}{1 - \epsilon} v_{23}\right).
$$

The right hand side is larger than u_1^{12} for ϵ small enough, again since $u_1^{12} < v_{13}$. This contradicts that $u_1^{12} > u_1^{13}$. Note that we have not used the fact that $v_{13} \geq v_{23}$. Thus, the same argument shows a contradiction if instead of $u_1^{12} < v_{13}$ we had assumed that $u_2^{12} < v_{23}$. In fact, we have not used the fact that $v_{12} > v_{13}, v_{23}$ but only the fact that $v_{12} < v_{13} + v_{23}$. Since this still happens when we make a permutation of the subindeces, we have shown that we cannot have as a limit $u_i^{ij} > u_i^{ij}$ and $u_j^{ij} > u_j^{jk}$ for any $i, j, k = 1, 2, 3$, $i \neq j \neq k$, $i \neq k$.

d.2) Second, assume that in the limit $u_1^{12} > u_1^{13}$ and $u_2^{12} < u_2^{23}$. According to the definition, this requires that for ϵ small $p_{13} \leq \epsilon$, and $p_{12} \leq \epsilon$. Thus, $p_{23} \geq 1 - 2\epsilon$. Also, $t_2^{12} = \frac{p_{23} u_2^{23}}{1-p_{12}}$ which converges to u_2^{23} and $t_1^{12} = \frac{p_{13} u_1^{13}}{1-p_{12}}$ which converges to 0. Thus, $t_1^{12} + t_2^{12}$ converges to U_2^{23}. Therefore, for ϵ

small either $t_1^{12} + t_2^{12} > v_{12}$ (and then $U_1^{12} = 0$), which would lead to a contradiction with $u_1^{12} > u_1^{13}$, or $u_2^{12} \geq t_2^{12}$, which contradicts $u_2^{12} < u_2^{23}$ since t_2^{12} converges to u_2^{23}. Again, note that we have not used the fact that $v_{12} > v_{13}, v_{23}$, and therefore we have just shown that we cannot have as a limit $u_i^{ij} > u_i^{ij}$ and $u_j^{ij} < u_j^{jk}$ for any $i, j, k = 1, 2, 3$, $i \neq j \neq k$, $i \neq k$.

d.3) Third, assume that in the limit $u_1^{12} > u_1^{13}$ and $u_2^{12} = u_2^{23}$. Note that we have already proved that it would be impossible to have also $u_3^{13} > u_3^{23}$. Thus we only need to consider $u_3^{13} < u_3^{23}$ and $u_3^{13} = u_3^{23}$. Thus, assume further that $u_3^{13} < u_3^{23}$. Then $p_{13} \leq \epsilon$. Also, assume that in the limit $p_{12} = 0$. In this case an argument similar to the one in d2) finds a contradiction: $t_2^{12} = \frac{p_{23} u_2^{23}}{1-p_{12}}$ which converges to u_2^{23} and $t_1^{12} = \frac{p_{13} u_1^{13}}{1-p_{12}}$ which converges to 0. Thus, $t_1^{12} + t_2^{12}$ converges to u_2^{23}. Therefore, for ϵ small either $t_1^{12} + t_2^{12} > v_{12}$, which would lead to a contradiction with $u_1^{12} > u_1^{13}$, or $u_2^{12} \geq t_2^{12}$. Since t_2^{12} converges to u_2^{23}, and in the limit $u_2^{12} = u_2^{23} = u_2$, then we conclude that in the limit $u_2^{12} = t_2^{12}$, which means that also in the limit $u_1^{12} = t_1^{12} = 0$, and this contradicts $u_1^{12} > u_1^{13}$. Thus, if $u_1^{12} > u_1^{13}$, $u_2^{12} = u_2^{23}$ and $u_3^{13} < u_3^{23}$, then $p_{12} > 0$ in the limit. Exactly the same argument, replacing $1, 2$ for $2, 3$ and vice versa, shows that in the limit $p_{23} > 0$. Thus, assume that $u_1^{12} > u_1^{13}$, $u_2^{12} = u_2^{23}$, $u_3^{13} < u_3^{23}$, $p_{12}, p_{23} > 0$, and $p_{13} = 0$ in the limit. Then, since $u_1^{12} > 0$ and $u_3^{23} > 0$, for ϵ small we have $t_2^{12} = \frac{p_{23} u_2^{23}}{1-p_{12}}$ which converges to $u_2^{23} = u_2^{12}$. Also, $t_2^{23} = \frac{p_{12} U_2^{12}}{1-p_{13}}$ which converges to $u_2^{12} = u_2^{23}$. This implies that $t_2^{12} + t_1^{12}$ converges to v_{12}, and since t_1^{12} converges to 0, u_2^{12} must converge to v_{12}. Likewise, U_2^{23} must converge to v_{23}. Thus, u_1^{12} and u_3^{23} converge to 0 and this again contradicts that in the limit $u_1^{12} > u_1^{13}$ and $u_3^{23} > u_3^{13}$. Therefore this case is not possible and the other case left to consider is $u_1^{12} > u_1^{13}$, $u_2^{12} = u_2^{23} = u_2$, $u_3^{13} = u_3^{23} = u_3$, and $p_{13} \leq \epsilon$ for ϵ small. Note that $u_1^{12} >$, and then for ϵ small, ad then in the limit, we must have that $u_1^{12} + u_2 = v_{12}$. Now, for $u_1^{12} = v_{12}$ in the limit, it would be required that for ϵ small t_1^{12} is sufficiently close to v_{12}. But $t_1^{12} = \frac{p_{13} u_1^{13}}{1-p_{12}} < \frac{p_{13} u_1^{12}}{1-p_{12}} \leq v_{12}$. The difference $u_1^{12} - u_1^{13}$ in the limit gurantees that this cannot infinitely approach v_{12}. Thus, we conclude that $u_2 > 0$, and therefore $u_2 + u_3 = v_{23}$. Therefore, for ϵ small

$$u_3 = \frac{1}{2}\left(v_{23} + t_3^{23} - t_2^{23}\right) = \frac{1}{2}\left(v_{23} + \frac{p_{13}}{p_{13}+p_{12}} u_3 - \frac{p_{12}}{p_{13}+p_{12}}(v_{23} - u_3)\right),$$

and solving for u_3 we obtain $u_3 = \frac{p_{13}}{p_{13}+p_{12}} v_{23}$. If $p_{12} > 0$ in the limit, this implies that in the limit $u_3 = v_{23}$ which implies that $u_2 = 0$. We have already seen that this leads to a contradiction. Therefore, it must be the case that $p_{12} = p_{13} = 0$ and $p_{23} = 1$ in the limit. Then in the limit $t_1^{12} = t_1^{13} = 0$, and $t_2^{12} = u_2 > 0$. This can only happen if for ϵ small u_2 is sufficiently close to v_{12}, so that in the limit $u_2 = v_{12}$ and therefore $u_1^{12} = 0$, which is a contradiction. Once again, we have not used the relative ranking of v_{ij} in this discussion, therefore this shows that we cannot have a limit with $u_i^{ij} > u_i^{ik}$ and $u_i^{ij} = u_j^{jk}$.

d.4) Thus, the only possibility left is that in equilibrium we have $u_1^{12} = u_1^{13} = u_1$, $u_2^{12} = u_2^{23} = u_2$, and $u_3^{13} = u_3^{23} = u_3$. Note that this implies that $t_i^{ij} \leq u_i$, for all $i, j = 1, 2$, $i \neq j$. Then not all these values u_i can be zero. Also, if two of the values were 0, say $u_1 = 0$, $u_2 = 0$, then for ϵ small $t_1^{12} + t_2^{12} < v_{12}$, which is a contradiction. Finally, if one of the values were 0, say $u_1 = 0$, but the rest are positive, that gurantees that for ϵ small $t_i^{ij} + t_j^{ij} \leq v_{ij}$ for all $i, j = 1, 2, 3$, $i \neq j$. Therefore, $u_2 = v_{12}$ and $u_3 = v_{13}$. But this is impossible, since $v_{12} + v_{13} > v_{23}$. The last inequality still holds if we reorder the subindexes, and therefore this shows that $u_i > 0$ for all $i = 1, 2, 3$. This is what we have considered in 5) above. We have seen there that this uniquely defines values u_i, and then also p and t are uniquely defined. This concludes the proof.

6.2 SpW: Optimal contract includes arbitrary large b

We compute the S's and E's payoffs for a given contract, (p, b), with $p \leq 1$, a given value of x and a realized value of y, assuming that payoffs in period 2 are determined by the $R-$solution. First, we obtain the game (N, v), as follows: $V = 1 - \min\{1-x, 1-y\} = \max\{x, y\}$; $v_B = 1 - p$, $v_E = 0$, $v_{SE} = p - \min\{1-x, 1-y\} = p + \max\{x, y\} - 1$.For the rest of the values, we have to distinguish between two cases: if $b > p + y - 1$, then $v_{BE} = 1 - p$ and $v_S = p + x - 1$; if $b \leq p + y - 1$, then $v_{BE} = y - b$ and $v_S = b$. Indeed, in the first case the pair BE would find it in their interest to buy from S, whereas in the second case they will prefer to trade themselves and pay the clause b. Given these values, we can compute Δ_{ij}, $i, j = B, S, E$, to determine the region in which we are, and also who is player $1, 2$, and 3, in our convention:

(i) If $b > p + y - 1$, then $\Delta_{BS} = \Delta_{BE} = 0$, and $\Delta_{SE} = \max\{y - x, 0\}$. Thus, we are in Region 1 of Section 2. (In this case, Region 1 and 2 coincide.)

(ii) If $b \leq p + y - 1$, $\Delta_{BS} = p + x - 1 - b$, $\Delta_{BE} = p + x - 1 - b$, and $\Delta_{SE} = p + \max\{x, y\} - 1 - b$. Thus, we are in Region 3 of Section 2.

In all cases, we can assign the index 1 to player S, the index 2 to player E, and the index 3 to player B.

Applying to these particular games the formulas provided in Table 1, we obtain that, in (i)

$$\begin{aligned} U_S &= p + \frac{x}{2} - 1 + \max\left\{\frac{x}{2}, \frac{y}{2}\right\}, \\ U_E &= \max\left\{0, \frac{y-x}{2}\right\}. \end{aligned}$$

Also, in (ii)

$$\begin{aligned} U_S &= \frac{x + 2\max\{x - y, 0\} + 2b + p - 1}{3}, \\ U_E &= \frac{y - b + p - 1}{3}. \end{aligned}$$

We can now compute the expected payoffs for players S and E, for a given value of x. If $b < p$, these payoffs are

$$\begin{aligned}\pi_S(x) &= \int_0^x (p+x-1)\, dH(y) + \\ &\int_x^{1+b-p} \left(p + \frac{x+y}{2} - 1\right) dH(y) + \int_{1+b-p}^1 \frac{x+2b+p-1}{3} dH(y),\end{aligned}$$

$$\pi_E(x) = \int_x^{1+b-p} \frac{y-x}{2} dH(y) + \int_{1+b-p}^1 \left(y - \frac{b+2x-p+1}{3}\right) dH(y),$$

if $x \leq 1+b-p$, and

$$\begin{aligned}\pi_S(x) &= \int_0^{1+b-p} (p+x-1)\, dH(y) + \\ &\int_{1+b-p}^x \left(x + \frac{2b-2y+p-1}{3}\right) dH(y) + \int_x^1 \frac{x+2b+p-1}{3} dH(y),\end{aligned}$$

$$\pi_E(x) = \int_{1+b-p}^x \frac{y-b+p-1}{3} dH(y) + \int_x^1 \left(y - \frac{b+2x-p+1}{3}\right) dH(y),$$

if $x > 1+b-p$. On the other hand, if $b \geq p$, these payoffs are

$$\pi_S(x) = \int_0^x (p+x-1)\, dH(y) + \int_x^1 \left(p + \frac{x+y}{2} - 1\right) dH(y),$$

$$\pi_E(x) = \int_x^1 \frac{y-x}{2} dH(y).$$

First, note that, for any x, $\pi_E(x)$ is lowest when $b \geq p$. Indeed, $\frac{y-x}{2} > y - \frac{b+2x-p+1}{3}$ when $y > x, 1+b-p$. That is, for every value of x the share of surplus that E appropriates is lowest if b is large enough. Even more, for given x, the sum of expected payoffs for E and B, that is $E[V] - \pi_E(x)$ are (weakly) increasing in b, since $V = \max\{x, y\}$ is independent of b for given x, and $\pi_E(x)$ is (weakly) decreasing in b. On the other hand, the incentives of S to invest, $\pi_S'(x)$, are largest when $b \geq p$. Indeed, in this case

$$\pi_S'(x) = \frac{1+H(x)}{2},$$

whereas if $b < p$, then

$$\pi_S'(x) = \frac{2+3H(x)+\max\{H(x), H(1+b-p)\}}{2}.$$

Lastly, the payoffs of B are (weakly) increasing in x. Indeed, if $b < p$, then

$$\pi_B(x) = \int_0^{1+b-p} (1-p)\, dH(y) + \int_{1+b-p}^1 \left(\frac{2(1-p)}{3} + \frac{\min\{x, y\} - b}{3}\right) dH(y),$$

and if $b \geq p$, these payoffs are

$$\pi_B(x) = \int_0^1 (1-p)\, dH(y).$$

Thus, the incentives of S to invest are (weakly) too low from the point of view of the pair SB, are highest when $b \geq p$, and, keeping x constant, the b that maximizes the joint payoffs of S and B is also any $b \geq p$. All this is independent of the value of p.Thus, we conclude that the optimal contract should include $b \geq p$.|

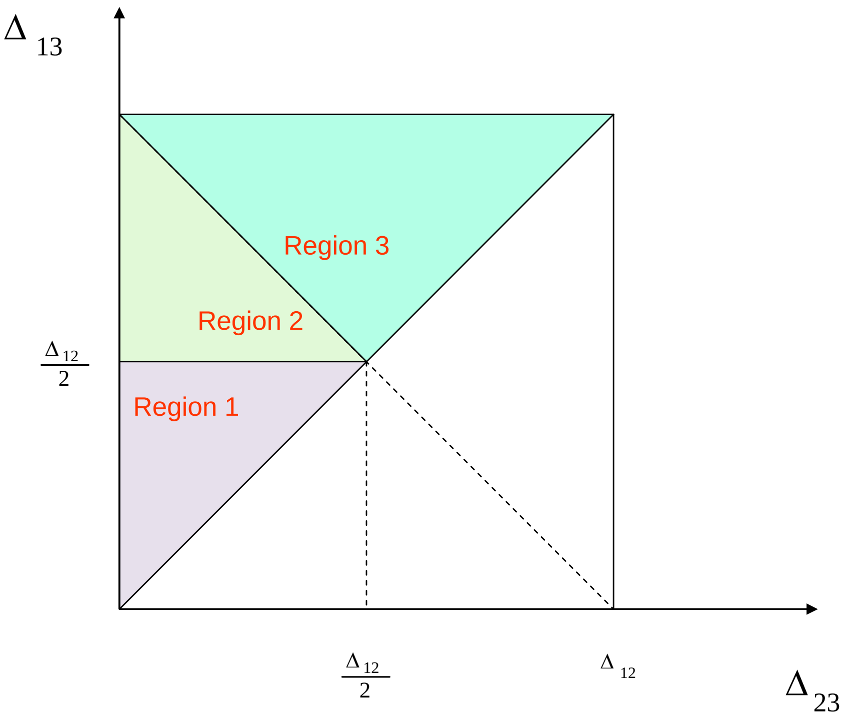
Δ_{13}
Region 3
Region 2
Region 1
$\frac{\Delta_{12}}{2}$
$\frac{\Delta_{12}}{2}$
Δ_{12}
Δ_{23}

Figure 1

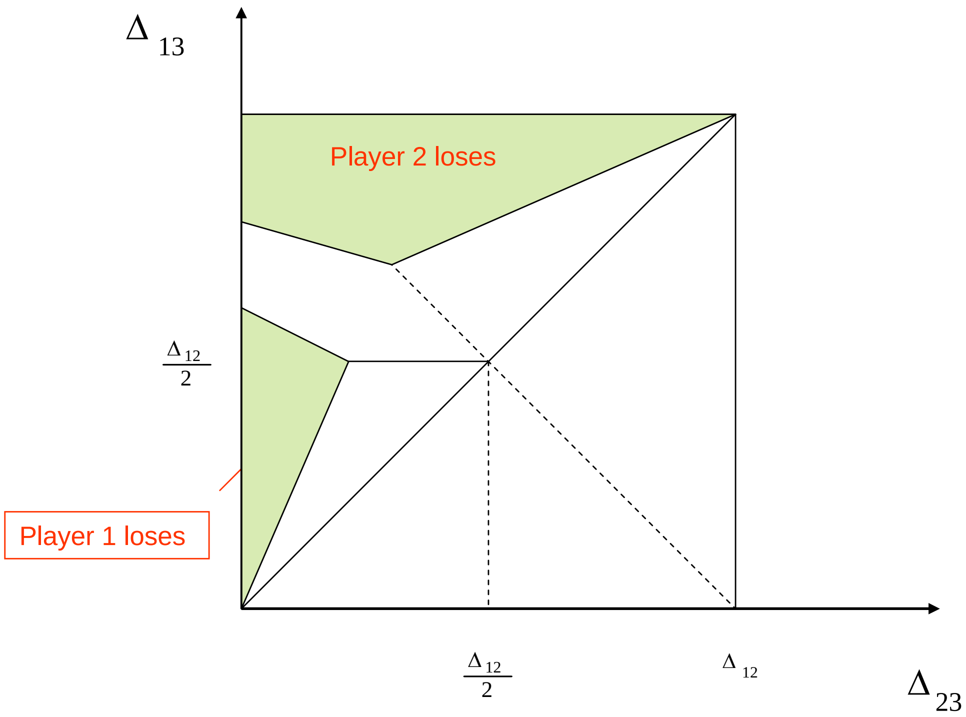

Figure 2